Discrete Mathematics 310 (2010) 3417–3420

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

Long cycles and paths in distance graphs

Lucia Draque Penso [a], Dieter Rautenbach [a,*], Jayme Luiz Szwarcfiter [b]

[a] *Institut für Optimierung und Operations Research, Universität Ulm, D-89069 Ulm, Germany*

[b] *Universidade Federal do Rio de Janeiro, Instituto de Matemática, NCE and COPPE, Caixa Postal 2324, 20001-970 Rio de Janeiro, RJ, Brazil*

ARTICLE INFO

Article history:
Received 23 November 2009
Received in revised form 19 July 2010
Accepted 20 July 2010
Available online 11 August 2010

Keywords:
Circulant graph
Distance graph
Connectivity
Hamiltonian cycle
Hamiltonian path

ABSTRACT

For $n \in \mathbb{N}$ and $D \subseteq \mathbb{N}$, the distance graph P_n^D has vertex set $\{0, 1, \dots, n-1\}$ and edge set $\{ij \mid 0 \le i, j \le n-1, |j-i| \in D\}$. Note that the important and very well-studied circulant graphs coincide with the regular distance graphs.

A fundamental result concerning circulant graphs is that for these graphs, a simple greatest common divisor condition, their connectivity, and the existence of a Hamiltonian cycle are all equivalent. Our main result suitably extends this equivalence to distance graphs. We prove that for a finite set D of order at least 2, there is a constant c_D such that the greatest common divisor of the integers in D is 1 if and only if for every n, P_n^D has a component of order at least $n - c_D$ if and only if for every $n \ge c_D + 3$, P_n^D has a cycle of order at least $n - c_D$. Furthermore, we discuss some consequences and variants of this result.

1. Introduction

Circulant graphs form an important and very well-studied class of graphs [2,17,18,22]. They are Cayley graphs of cyclic groups and have been proposed for numerous network applications such as local area computer networks, large area communication networks, parallel processing architectures, distributed computing, and VLSI design. Their connectivity and diameter [5,2,17,18], cycle and path structure [1,3,4,6], and further graph-theoretical properties have been studied in great detail. Polynomial time algorithms for isomorphism testing and recognition of circulant graphs have been long-standing open problems which were completely solved only recently [15,21].

For $n \in \mathbb{N}$ and $D \subseteq \mathbb{N}$, the *circulant graph* C_n^D has vertex set $[0, n-1] = \{0, 1, \dots, n-1\}$ and the neighbourhood $N_{C_n^D}(i)$ of a vertex $i \in [0, n-1]$ in C_n^D is given by

$$N_{C_n^D}(i) = \{(i+d) \bmod n \mid d \in D\} \cup \{(i-d) \bmod n \mid d \in D\}.$$

Clearly, we may assume $\max(D) \le \frac{n}{2}$ for every circulant graph C_n^D.

Our goal here is to extend a fundamental result concerning circulant graphs to the similarly defined yet more general class of distance graphs: For $n \in \mathbb{N}$ and $D \subseteq \mathbb{N}$, the *distance graph* P_n^D has vertex set $[0, n-1]$ and

$$N_{P_n^D}(i) = \{i+d \mid d \in D \text{ and } (i+d) \in [0, n-1]\} \cup \{i-d \mid d \in D \text{ and } (i-d) \in [0, n-1]\}$$

for all $i \in [0, n-1]$. Clearly, we may assume $\max(D) \le n-1$ for every distance graph P_n^D.

Every distance graph P_n^D is an induced subgraph of the circulant graph $C_{n+\max(D)}^D$. More specifically, distance graphs are the subgraphs of sufficiently large circulant graphs induced by sets of consecutive vertices. Conversely, our following simple observation from [11] shows that every circulant graph is in fact a distance graph.

Proposition 1 ([11]). *A graph is a circulant graph if and only if it is a regular distance graph.*

* Corresponding author.
E-mail addresses: lucia.penso@uni-ulm.de (L.D. Penso), dieter.rautenbach@uni-ulm.de, dieter.rautenbach@tu-ilmenau.de (D. Rautenbach), jayme@nce.ufrj.br (J.L. Szwarcfiter).

0012-365X/$ – see front matter
doi:10.1016/j.disc.2010.07.020

Proof. Clearly, every circulant graph C_n^D is regular and isomorphic to the distance graph $P_n^{D'}$ for $D' = D \cup \{n - d \mid d \in D\}$.

Now let P_n^D be a regular distance graph. Let $D = \{d_1, d_2, \ldots, d_k\}$ with $d_1 < d_2 < \cdots < d_k \leq n - 1$. Since the vertex 0 has exactly k neighbours D, P_n^D is k-regular.

Let $i \in [1, k]$. The vertex $d_i - 1$ has exactly $i - 1$ neighbours j with $j < d_i - 1$. Hence, $d_i - 1$ has exactly $k + 1 - i$ neighbours j with $j > d_i - 1$ which implies $(d_i - 1) + d_{k+1-i} \leq n - 1$. The vertex d_i has exactly i neighbours j with $j < d_i$. Hence, d_i has exactly $k - i$ neighbours j with $j > d_i$ which implies $d_i + d_{k+1-i} > n - 1$.

We obtain $d_i + d_{k+1-i} = n$ for every $i \in [1, k]$ which immediately implies that P_n^D is isomorphic to the circulant graph $C_n^{D'}$ for $D' = \{d \in D \mid d \leq \frac{n}{2}\}$. $\square$

Distance graphs lack the symmetry and algebraic interpretation of circulant graphs and the algebraic methods used in [15,21] will not apply to them. In view of Proposition 1, the recognition of distance graphs will be at least as difficult as the recognition of circulant graphs.

Originally, coloring problems for infinite distance graphs were studied by Eggleton et al. [13,14]. In fact, most research on distance graphs focused on colorings [8,10,12,19,23].

One of the most fundamental results for circulant graphs is the following beautiful equivalence.

Theorem 2 ([5,7,16]). *For $n \in \mathbb{N}$ and a finite set $D \subseteq \mathbb{N}$, the following statements are equivalent.*

(i) *C_n^D is connected.*
(ii) *The greatest common divisor $\gcd(\{n\} \cup D)$ of the integers in $\{n\} \cup D$ equals 1.*
(iii) *C_n^D has a Hamiltonian cycle.*

In 1970 Lovász [9,20,24] asked whether every connected vertex-transitive graph has a Hamiltonian path. Since circulant graphs are clearly vertex transitive, Theorem 2 is a positive example for this well-studied problem [9,24].

In the present paper, we suitably extend Theorem 2 to distance graphs. While connectivity and hamiltonicity of circulants are equivalent to a simple necessary gcd-condition, we prove that a similar condition for distance graphs is only equivalent to the existence of a large component and a long cycle. We also discuss consequences and variants of our result.

2. Cycles and paths in distance graphs

We immediately proceed to our main result. The residue of an integer $n \in \mathbb{Z}$ modulo $d \in \mathbb{N}$ will be denoted by $n \bmod d$.

Theorem 3. *For a finite set $D \subseteq \mathbb{N}$ with $|D| \geq 2$, the following statements are equivalent.*

(i) *There is a constant $c_1(D)$ such that for every $n \in \mathbb{N}$, the distance graph P_n^D has a component of order at least $n - c_1(D)$.*
(ii) $\gcd(D) = 1$.
(iii) *There is a constant $c_2(D)$ such that for every $n \in \mathbb{N}$ with $n \geq c_2(D) + 3$, the distance graph P_n^D has a cycle of order at least $n - c_2(D)$.*

Proof. (i) $\Rightarrow$ (ii): Let n be such that n is even and $n > 2c_1(D)$. By (i), more than half the vertices are in the same component of P_n^D. By the pigeonhole principle, there is some $i \in [0, n-2]$ such that the two vertices i and $i+1$ are in the same component of P_n^D. This implies that there is a path in P_n^D from i to $i + 1$. Hence 1 is an integral linear combination of the elements in D. It is a well-known consequence of the Euclidean algorithm that this is equivalent to (ii).

(ii) $\Rightarrow$ (iii): Let $d = \max(D)$. Let $\vec{C}_d^{D\setminus\{d\}}$ denote the directed circulant graph with vertex set $[0, d - 1]$ where (i, j) is an arc for $i, j \in [0, d - 1]$ if and only if $(j - i) \bmod d \in D \setminus \{d\}$.

Claim. *$\vec{C}_d^{D\setminus\{d\}}$ has a Hamiltonian path.*

Proof of Claim. We prove the statement by induction on $|D|$. For $|D| = 2$, let $D = \{d, d'\}$. Since $\gcd(d, d') = 1$, we have $\{id' \bmod d \mid 0 \leq i \leq d - 1\} = [0, d - 1]$, i.e.

$$0(d' \bmod d)(2d' \bmod d)(3d' \bmod d) \ldots ((d - 1)d' \bmod d)$$

is a Hamiltonian path. Now let $|D| > 2$. Let $D' = D \setminus \{d'\}$ for some $d' \in D \setminus \{d\}$. Let $\gcd(D') = g$. By induction, $\vec{C}_d^{D'\setminus\{d\}}$ has a path P which visits exactly all vertices in $[0, d - 1]$ which are multiples of g. Clearly, we may assume that $g \geq 2$. Since $\gcd(g, d') = 1$, we have $\{id' \bmod g \mid 0 \leq i \leq g - 1\} = [0, g - 1]$ and concatenating translates of P and arcs of the form $(i, i + d')$ yields a Hamiltonian path of $\vec{C}_d^{D\setminus\{d\}}$ as desired. $\square$

By the claim, there are $a_1, a_2, \ldots, a_{d-1} \in D \setminus \{d\}$ such that

$$0(a_1 \bmod d)((a_1 + a_2) \bmod d)((a_1 + a_2 + a_3) \bmod d) \cdots ((a_1 + a_2 + \cdots + a_{d-1}) \bmod d)$$

is a Hamiltonian path of $\vec{C}_d^{D\setminus\{d\}}$, i.e. $\left\{\left(\sum_{i=1}^k a_i\right) \bmod d \mid 0 \leq k \leq d - 1\right\} = [0, d-1]$. Let $A = a_1 + a_2 + \cdots + a_{d-1} \leq d(d-1)$. Clearly, we may assume that n is sufficiently large in terms of D. If $m \in \mathbb{N}$ is such that $n - 2d \leq (2m - 1)d + A < n$, then

$$\left\{ld + \sum_{i=1}^k a_i \mid l \in [0, 2m - 1], k \in [0, d - 1]\right\}$$

is a set of $2md$ distinct vertices which induce a subgraph of P_n^D that is isomorphic to $P_{2m} \times P_d$. Since $P_{2m} \times P_d$ obviously has a Hamiltonian cycle, P_n^D has a cycle of order at least $2md \geq n - d - A \geq n - d^2$, i.e. (iii) holds.
(iii) $\Rightarrow$ (i): Since this implication is trivial, the proof is complete. □

We add some comments concerning Theorem 3.

It is easy to see that a distance graph P_n^D with $\gcd(D) = 1$ and $n \geq 2\max(D) + 1$ is actually connected. Hence (i) in Theorem 3 could be replaced by

(i)′ *There is a constant $c_3(D)$ such that for every $n \in \mathbb{N}$ with $n \geq c_3(D)$, the distance graph P_n^D is connected.*

For (iii) in Theorem 3, a similar change is not possible, because no lower bound on the order n would imply that P_n^D has a Hamltonian cycle. If n as well as all elements of D are odd for instance, then P_n^D is bipartite and every cycle misses at least one vertex. In this sense, Theorem 3 is best possible. It follows easily from our proof that $c_2(D) = O(\max(D)^2)$. For the case that $\gcd(D)$ is different from 1, Theorem 3 implies the following corollary.

Corollary 4. *For a finite set $D \subseteq \mathbb{N}$ with $|D| \geq 2$ and $g \in \mathbb{N}$, the following statements are equivalent.*

(i) *There is a constant $c_4(D)$ such that for every $n \in \mathbb{N}$, the distance graph P_n^D has a component of order at least $\frac{n}{g} - c_4(D)$.*
(ii) $\gcd(D) \leq g$.
(iii) *There is a constant $c_5(D)$ such that for every $n \in \mathbb{N}$ with $n \geq g(c_2(D) + 3)$, the distance graph P_n^D has a cycle of order at least $\frac{n}{g} - c_5(D)$.*

Theorem 3 trivially implies yet another condition which is equivalent to (i), (ii), and (iii) in Theorem 3.

(iv) *There is a constant $c_6(D)$ such that for every $n \in \mathbb{N}$, the distance graph P_n^D has a path of order at least $n - c_6(D)$.*

Clearly, such a path can be obtained from the cycle in (iii) by deleting one edge. In fact, if $v_{(l,k)} = ld + \sum_{i=1}^{k} a_i$ denotes the vertices of the induced subgraph $P_{2m} \times P_d$ of P_n^D identified in the proof of "(ii) $\Rightarrow$ (iii)" of Theorem 3, then

$$v_{(0,0)}v_{(0,1)}\cdots v_{(0,d-1)}v_{(1,d-1)}v_{(1,d-2)}\cdots v_{(1,0)}v_{(2,0)}v_{(2,1)}\cdots v_{(2,d-1)}v_{(3,d-1)}v_{(3,d-2)}\cdots v_{(3,0)}\cdots v_{(2m-1,d-1)}$$

specifies a Hamiltonian path of $P_{2m} \times P_d$ which proves the following.

Corollary 5. *For a finite set $D \subseteq \mathbb{N}$ with $|D| \geq 2$, the following statements are equivalent.*

(i) $\gcd(D) = 1$.
(ii) *There are two constants $c_7(D)$ and $c_8(D)$ such that for every $n \in \mathbb{N}$, the distance graph P_n^D has a path $u_0u_1 \ldots u_l$ of order at least $n - c_7(D)$ such that $u_j > u_i$ for all $0 \leq i, j \leq l$ with $j - i \geq c_8(D)$.*

We believe that Corollary 5 can be strengthened as follows.

Conjecture 6. *For a finite set $D \subseteq \mathbb{N}$ with $|D| \geq 2$ and $\gcd(D) = 1$, there is some $n \in \mathbb{N}$ such that $n \geq 2$ and P_n^D has a Hamiltonian path with endvertices 0 and $n - 1$.*

As our last result, we verify Conjecture 6 for $|D| = 2$.

Proposition 7. *If $d_1, d_2 \in \mathbb{N}$ are such that $d_1 > d_2$ and $\gcd(\{d_1, d_2\}) = 1$, then $P_{d_1+d_2+1}^{\{d_1,d_2\}}$ has a Hamiltonian path with endvertices 0 and $d_1 + d_2$.*

Proof. Consider the sequence $i_0, i_1, \ldots, i_{d_1+d_2}$ produced by Algorithm 1.

```
i_0 := 0;
n := 0;
n_1 := 0;
n_2 := 0;
while n < d_1 + d_2 do
    if i_n >= d_2 and n_2 < d_1 - 1 then
        i_{n+1} := i_n - d_2;
        n_2 := n_2 + 1;
    else
        i_{n+1} := i_n + d_1;
        n_1 := n_1 + 1;
    end
    n := n + 1;
end
```

Algorithm 1

Clearly, $i_j \geq 0$ for $j \in [0, d_1 + d_2]$.
If $n_1 > d_2 + 1$ after the termination of the algorithm, then $n_2 = n - n_1 < d_1 - 1$ and hence

$$i_{d_1+d_2} = n_1d_1 - n_2d_2 \geq (d_2 + 2)d_1 - (d_1 - 2)d_2 = 2d_1 + 2d_2.$$

Let $j \in [1, d_1+d_2-1]$ be maximum such that $i_{j+1} = i_j+d_1$. Clearly, $i_{j+1} \geq d_2$ and the algorithm would have set $i_{j+1} = i_j - d_2$ instead, which is a contradiction. Hence, $n_1 \leq d_2 + 1$. Since $n_2 \leq d_1 - 1$, we obtain $n_1 = d_2 + 1$ and $n_2 = d_1 - 1$. This implies

$$i_{d_1+d_2} = (d_2 + 1)d_1 - (d_1 - 1)d_2 = d_1 + d_2.$$

If $i_j > d_1 + d_2$ for some $j \in [1, d_1 + d_2 - 1]$, then let j be largest with this property. Clearly, $i_{j-1} \geq d_2$ and at this moment of the execution of the algorithm $n_2 < d_1 - 1$. Therefore, the algorithm would have set $i_j = i_{j-1} - d_2$ instead, which is a contradiction. Hence, $i_j \leq d_1 + d_2$ for all $j \in [0, d_1 + d_2]$.

If $i_r = i_s$ for some $r, s \in [0, d_1 + d_2]$ with $s > r$, then $i_s - i_r = a_1d_1 - a_2d_2 = 0$ for some $a_2 \in [1, d_1 - 1]$. This implies $a_1d_1 = a_2d_2$. Since $\gcd(\{d_1, d_2\}) = 1$, we obtain that a_2 must be a multiple of d_1, which is a contradiction. Hence, all $d_1 + d_2 + 1$ integers $i_0, i_1, \ldots, i_{d_1+d_2}$ are distinct and define the desired Hamiltonian path of $P^{\{d_1,d_2\}}_{d_1+d_2+1}$. This completes the proof. □

References

[1] T. Araki, Y. Shibata, Pancyclicity of recursive circulant graphs, Inform. Process. Lett. 81 (2002) 187–190.
[2] J.-C. Bermond, F. Comellas, D.F. Hsu, Distributed loop computer networks: a survey, J. Parallel Distrib. Comput. 24 (1985) 2–10.
[3] J.C. Bermond, O. Favaron, M. Maheo, Hamiltonian decomposition of Cayley graph of degree four, J. Combin. Theory Ser. B 46 (1989) 142–153.
[4] D.K. Biss, Hamiltonian decomposition of recursive circulant graphs, Discrete Math. 214 (2000) 89–99.
[5] F. Boesch, R. Tindell, Circulants and their connectivities, J. Graph Theory 8 (1984) 487–499.
[6] Z.R. Bogdanowicz, Pancyclicity of connected circulant graphs, J. Graph Theory 22 (1996) 167–174.
[7] R.E. Burkard, W. Sandholzer, Efficiently solvable special cases of bottleneck travelling salesman problems, Discrete Appl. Math. 32 (1991) 61–76.
[8] G.J. Chang, L. Huang, X. Zhu, Circular chromatic numbers and fractional chromatic numbers of distance graphs, European J. Combin. 19 (1998) 423–431.
[9] S.J. Curran, J.A. Gallian, Hamiltonian cycles and paths in Cayley graphs and digraphs—a survey, Discrete Math. 156 (1996) 1–18.
[10] W. Deuber, X. Zhu, The chromatic number of distance graphs, Discrete Math. 165–166 (1997) 195–204.
[11] L. Draque Penso, D. Rautenbach, J.L. Szwarcfiter, Connectivity and diameter in distance graphs, Manuscript, 2009.
[12] B. Effantin, H. Kheddouci, The *b*-chromatic number of some power graphs, Discrete Math. Theor. Comput. Sci. 6 (2003) 45–54.
[13] R.B. Eggleton, P. Erdős, D.K. Skilton, Coloring the real line, J. Combin. Theory Ser. B 39 (1985) 86–100.
[14] R.B. Eggleton, P. Erdős, D.K. Skilton, Colouring prime distance graphs, Graphs Combin. 6 (1990) 17–32.
[15] S.A. Evdokimov, I.N. Ponomarenko, Circulant graphs: recognizing and isomorphism testing in polynomial time, St. Petersbg. Math. J. 15 (2004) 813–835. (English. Russian original), Translation from Algebra Anal. 15 (2003) 1–34.
[16] R.S. Garfinkel, Minimizing wallpaper waste, part I: a class of traveling salesman problems, Oper. Res. 25 (1977) 741–751.
[17] F.K. Hwang, A complementary survey on double-loop networks, Theoret. Comput. Sci. 263 (2001) 211–229.
[18] F.K. Hwang, A survey on multi-loop networks, Theoret. Comput. Sci. 299 (2003) 107–121.
[19] A. Kemnitz, M. Marangio, Colorings and list colorings of integer distance graphs, Congr. Numer. 151 (2001) 75–84.
[20] L. Lovász, Problem 11, in: Combinatorial Structures and their Applications, University of Calgary, Calgary, Alberta, Canada, Gordon and Breach, New York, 1970.
[21] M. Muzychuk, A solution of the isomorphism problem for circulant graphs, Proc. Lond. Math. Soc. (3) 88 (2004) 1–41.
[22] C.S. Raghavendra, J.A. Sylvester, A survey of multi-connected loop topologies for local computer networks, Comput. Netw. ISDN Syst. 11 (1986) 29–42.
[23] M. Voigt, Colouring of distance graphs, Ars Combin. 52 (1999) 3–12.
[24] D. Witte, J. Gallian, A survey: Hamiltonian cycles in Cayley graphs, Discrete Math. 51 (1984) 293–304.